Mental computation: An opportunity to develop students' strategies in rational number division

Renata Carvalho[1] and João Pedro da Ponte[2]

[1]Instituto de Educação, Universidade de Lisboa, Portugal; renatacarvalho@campus.ul.pt

[2]Instituto de Educação, Universidade de Lisboa, Portugal; jpponte@ie.ulisboa.pt

Mental computation provides a good environment to develop students' strategies in division of rational numbers. In this paper, we seek to analyze mental computations strategies in rational number division used by grade 6 students, during a teaching experiment. The teaching experiment emphasizes collective discussions and was based on tasks involving number sentences and word problems with the four basic operations. The methodology was design-based research with two experimental cycles involving two teachers and 39 students. All lessons were audio and video recorded. The results show that students mostly use numerical relationships strategies supported by propositional representations in rational number division. Initially, students' strategies are based on applying a procedure and evolve to strategies that relate division and multiplication.

Keywords: Mental computation, students' strategies, rational numbers, division.

Introduction

Learning mathematics is not just about memorizing and replicating a set of procedures. A deep understanding of concepts is needed to learn mathematics with meaning and to develop mathematical proficiency. The development of mathematical proficiency stands on the interrelation between five strands: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning and productive dispositions (Kilpatrick, Swafford, & Findell, 2001). Procedural fluency needs to be built based on conceptual understanding. The development of students' mental computation skills with rational numbers, using word problems and number sentences, can provide a good environment to achieve such conceptual understanding. The discussion of mental computation strategies allows teachers to understand students' thinking about rational numbers, their conceptual understanding. Furthermore, it helps students to construct and reconstruct new knowledge based on what they previously learned. In this paper, we seek to analyze what mental computation strategies grade 6 students use in rational numbers division during a teaching experiment. The teaching experiment emphasizes collective discussions and was based on mental computation tasks involving number sentences and word problems with the four operations

Division of rational numbers

For students, division is a difficult operation. It begins with whole numbers and becomes more complicated with fractions and decimals. According to Sinicrope, Mick and Kolb (2002), division has different meanings that students need to learn and understand. In fact, division can be used to: (a) determine the number of groups (measurement division – when dividing $2 \div 1/2$, we find the number of times $1/2$ falls within 2); (b) determine the size of each group (partitive division – when dividing $3/4$ of a pie equally by 3 people, we find the amount of pie for each person); (c) determine the dimension of a rectangle array (inverse of a Cartesian product – we find one dimension, in a problem of area in which the total area and other dimension are known). When working only with fractions, the authors add two more meanings: (d) to determine a unit rate (a printer can print 100 pages in two and one-half minutes. How many pages does it prints in one minute?), and (e) as the inverse of multiplication. A deep understanding of rational number division involves understanding all these meanings, but, Siebert (2002) emphasizes the key importance of measurement and partitive division. Usually, students use the "invert and multiply" rule to divide fractions focusing only in procedures. For this author, this algorithm does not seem to be associated with division, since it has no signs of division and it is not in line with students' understanding of what division means (finding the number of groups or finding the size of each group). Sinicrope et al. (2002) consider these meanings important, but highlight the importance of connecting the context of a problem with the algorithm to be used, since each context triggers a set of procedures to solve a problem. In line with this perspective, Siebert (2002) stresses that division with fractions must be explored using real-life contexts to help students to create images and make connections between the solutions of these problems and the knowledge they have about division with whole numbers, as this allows students to extend their knowledge. After constructing these images with meaning, they are prepared to understand the "invert and multiply" rule.

Mental computation with rational numbers

In this study, mental computation is seen as an exact computation made mentally in a quick and effective way, using mental representations and involving number facts, memorized rules, and relationships between numbers and operations (Carvalho & Ponte, 2017). In addition, students need to understand the size and value of numbers and the effect of an operation on a number, as well as to be able to make estimates to check the reasonableness of solutions (Heirdsfield, 2011). When computing with rational numbers, notably with fractions, a reconceptualization is needed because, multiplying and dividing fractions not always produces a product bigger that the factors or a quotient smaller that the dividend. As computing with rational numbers involves more complex reasoning than computing with whole numbers, we assume that the use of memorized rules (e.g., application of procedures such as multiplying/dividing by powers of 10) may sometimes support students' computation and the establishment of numerical relationships. Number facts used by students in mental computation can involve knowledge about results of some operations (sums, differences, products, and quotients) or relationships among numbers and operations that they have stored in their memory throughout school experiences. Using numerical relationships, involves a deep understanding of numbers and operations, the capacity to use fundamental properties of operations and the notion of equality to analyse and solve problems (Empson, Levi, & Carpenter, 2010). Some numerical relationships strategies used by students are related to the change of rational number representations (Carvalho & Ponte, 2017) (e.g., fraction→decimal; decimal→fraction or a rational number to a whole number concerning 10/100), to equivalences between mathematical expressions and to inverse relationship between operations. Number facts and memorized rules can emerge as mental computation strategies, *per se*, but they also arise as a support in establishing relationships between numbers and operations and vice-versa.

In the perspective of Dehaene (1997), memory plays a central role in mental computation, not only for its ability to store number facts, but also for the mental models that it creates, based on previous knowledge, supporting students' reasoning. In mental computation we use mental representations from the world that surrounds us, in sense making and in making inferences. According to the theory of mental models (Johnson-Laird, 1990), such mental representations may be: (i) mental models, if they are general perceptions of the world (e.g., using a general context of fair sharing to share a given quantity); (ii) mental images, if they involve a more specific perception of the real world where some characteristics are considered (e.g., relating the symbolic representation $1/2$ to a pizza divided in two parts taking only one part); and, (iii) propositional representations, if they represent true or false statements that play an important role in the inference process (e.g., to compute $1/4 \div ? = 1/2$, we realize that if $1/2 \times 1/2 = 1/4$ then the missing value is $1/2$).

Research methodology

This study is qualitative and interpretative with a design research approach (Cobb et al., 2003). As a developmental study, it seeks to solve problems identified from practice, mainly the difficulties in learning rational numbers and near absence of mental computation with this number set. It bases on a teaching experiment with mental computation tasks that provide opportunities for discussing students' strategies. It includes three phases: preparation; experimentation and analysis. In the preparation phase (2010) we undertook a preliminary study in grade 5 (conducted by the first author in her classes) to understand students' strategies when computing mentally with rational numbers, and to figure out practical aspects of students' mental computation strategies, important for planning the teaching experiment. Such planning also considered the research on rational numbers and mental computation with rational numbers. In the second phase (2012-13), cycle 1 (CI) and 2 (CII) were carried out involving two teachers and two grade 6 classes (39 students) from two different schools selected according to teacher's availability to participate in the study, with the first author as a participant observer. During this phase some refinements were made in the teaching experiment (e.g., changing the sequence of tasks). Data were collected through video and audio recordings of the classroom work with mental computation. Finally, in the analysis phase, the dialogues recording students' strategies were transcribed. In this paper we will focus on students' strategies to compute with division of rational numbers with fractions and decimals. To analyze data, we defined three main categories of strategies and several subcategories (Table 1) that we relate with students' mental representations (images, models and propositional representations). These categories emerged from the data informed by the literature review.

	Categories		
	Numerical relationships	Number facts	Memorized rules
Subcategories	Change representation	Two halves make a unit	Rule to add/subtract fractions
	Part-whole comparison	A half of a half is a quarter	Invert and multiply rule
	Relation between operations		

Table 1. Categorization of mental computation strategies with rational numbers

The categorization was made according to the strongest notion involved in students' strategy. For example, if there is a strong use of numerical relationships, as the change of representation, we coded this strategy in the category "numerical relationships" and subcategory "change representation".

The teaching experiment

The teaching experiment relies on the conjecture that a systematic work with mental computation tasks with rational numbers represented as fractions, decimals and percent, and whole class discussions may contribute to the development of students' mental computation strategies and understanding of their errors. Before the teaching experiment, the students already had worked with rational numbers in different representations and operations, with emphasis on algorithms. The teaching experiment included ten mental computation tasks with rational numbers (with several number sentences or word problems each) to carry out weekly All tasks were provided by the first author and discussed in detail with the participating teachers. These tasks were presented at the beginning of a class by using a timed PowerPoint to challenge students to compute mentally in a faster way. The students had 15 seconds to solve each number sentence and 20 seconds to solve each word problem individually. The results were recorded on paper. Each task has two parts, both with 5 number sentences or 4 word problems. Upon finishing each part, there was a collective discussion of students' strategies. These mental computation lessons lasted 30 to 90 minutes. The discussion moments were regarded as very important. They allowed students to show how they think and which strategies they use to compute mentally in tasks with different cognitive demands. These moments were important for students to think, reflect, analyze, make connections, share, and extend mental computation strategies, as well as to identify skills that they should develop about numbers and operations.

The students began to compute mentally with fractions (addition/subtraction in task 1 and multiplication/ division in task 2), then with decimals and fractions with the four basic operations (task 3), and then only with decimals (addition/subtraction in task 4 and multiplication/division in task 5). Subsequently, they solved word problems in measurement and comparison contexts involving fractions and decimals (task 6). Percent was used in task 7, as the teacher begun working with statistics. Then, students used decimals, fractions and percent in tasks 8, 9 and 10. In task 10 they solved word problems. The tasks were designed following three principles and considering previous research on mental computation and rational numbers: *Principle 1*. Use contexts to help students to give meaning to numbers. A structured knowledge is associated with the context in which it was learned and, most of the time, it is difficult for a student to bridge this knowledge to new situations. *Principle 2*. Use multiple representations of rational numbers. We used fractions, decimals, and percent representations in the same task and in several tasks along the teaching experiment (e.g., $2{,}4 \div 1/2$). We used even numbers and multiples of 5 and 10, benchmarks such as 25% or 1/2 to facilitate equivalence between decimals, fractions and percent, and to stress numerical and part-whole relationships. *Principle 3*. Use tasks with different levels of cognitive demand. For example, taking into account mental computation levels (Callingham & Watson, 2004) we designed tasks in which the students have to use the concept of half (e.g. 50% of 20 or 1/2+1/2) or need to use a more complex numerical relationships (e.g., 20% of ? = 8), to do the computation. When planning the lessons, we sought to anticipate students' possible strategies to better prepare the collective discussions. All classroom activities were led by the teachers, with the first author making occasional interventions to ask students to clarify their strategies.

Students' metal computation strategies for division of rational numbers

In this section we analyse students' mental computation strategies in dividing rational numbers with fractions and decimals. The examples selected (Table 2) are representative for the strategies used by students in CI and CII. We begin with questions using division by 1/2 because we think this is an essential step to understand the division of rational numbers, especially the relation between dividend, divisor and quotient. We used other benchmarks as 1/4 or the division between a fraction and a whole number to support students' extension of knowledge from the division of whole numbers.

Mental computation question		Mental computation strategies	Mental representation	Students' meaning of division
Task 2	$\frac{4}{8} \div \frac{1}{2} = ?$	**Rita (C I)**: *Memorized rules* (invert and multiply rule)	*Mental image*	*Application of a procedure*
	$\frac{1}{4} \div ? = \frac{1}{2}$	**Ana (C I)**: *Numerical relationship* (relation between expressions- multiplication and division*) and number facts* (half of a half is a quarter)	*Propositional representation*	*Inverse relationship between multiplication and division*
	$\frac{3}{4} \div ? = \frac{1}{4}$	**António (C II):** *Numerical relationship* (relation between expressions- division/multiplication/ repeat addition*) and number facts* (quotient known)	*Mental model Propositional representation*	*Partitive division*
Task 5	$2.1 \div ? = 8.4$	**Maria (C I)**: *Numerical relationship* (relation between numbers and expressions- division/multiplication) *and number facts* (time table)	*Propositional representation*	*Inverse relationship between multiplication and division*
Task 6	*A tank has the capacity of 22.5 l. How many 1/2 l buckets do you need to completely fill the tank?*	**Eva (C I)**: *Numerical relationship* (relation between operations - ½ and the multiplication by 2; part-whole relationship)	*Propositional representation*	*Measurement division*
Task 10	$0.75 \div ? = 3$	**Rui (C II)**: *Numerical relationship* (change representation decimal→fraction; relation between numbers)	*Propositional representation*	*Inverse relationship between multiplication and division*

Table 2: Synthesis of Students strategies in rational numbers division

In task 2, students compute mentally with fractions in number sentences and open number sentences. To solve "$4/8 \div 1/2 = ?$" students could simply identify that dividend and divisor represent equivalent fractions, so the quotient is 1. However, most of the students apply the "invert and multiply" rule as Rita did: "I inverted the 2 with the 1. I did 4 times 2 and get 8 and 8 times 1 and get 8. I get the unit. I wrote 1". Rita used a *memorized rule* probably supported by *mental images* of procedures that she knows. In the 2nd part of task 2, Ana's strategy to compute "$1/4 \div ? = 1/2$" and António's strategy to calculate "3/4÷?=1/4" led to discuss another way of thinking about dividing rational numbers. Ana explained: "I just know that 1/2 times 1/2 is 1/4. So, I put immediately 1/2". She used a *numerical relationship* strategy (relation between expressions) based on a *propositional representation* (if 1/2×1/2=1/4 so1/4÷1/2=1/2) supported by a *number fact* (half of a half is a quarter). Her strategy emphasizes the relationship between multiplication and division as inverse operations. To compute “3/4÷?=1/4” António explained: “I thought of 3€. I forgot the 4 [in the denominator]. 3€ dividing by 3 persons is equal 1€ to each person… 3/4 dividing by 3 persons is 1/4 . . . I thought 1/4+1/4+1/4 [is] 3/4”. António used a *numerical relationship* strategy (relations between expressions). He relates division (3/4÷3) with multiplication through repeated addition (3×1/4=1/4+1/4+1/4). António uses the context of money (*mental model*) to share money between 3 persons, and so, he “forgot the 4” from the denominator to make an extension of what he knows about whole numbers division. He probably searched in his quotients repertoire, one that gives 1 (*number facts*), and validated his strategy using a *propositional representation* (if 3/4÷?=1/4 and 3÷3=1, so, ?=1/4 because 1/4+1/4+1/4=3/4). António’s strategy shows how a partitive division context that he knows can be a model to think and compute the given mathematical expression.

In task 5, Maria explained her strategy to compute “2.1÷?=8.4”, showing flexibility in working with equivalent representations of rational numbers:” I did 2 times 2.1 and then 4 times and I get it [8.4]. But we must divide. The inverse of 4 is 1/4 that is 0.25”. She used a *numerical relationship* strategy (relation between numbers and operations) supported by *number facts* and *propositional representations*. To find the missing number, Maria used *number facts* (time table of 2 and 4) to find the multiplicative relationship between the 2.1 and 8.4. The given operation was division and she used multiplication (inverse operation) to think. So, she needs to answer using the “inverse of 4 is 1/4 that is 0.25”. Her strategy could be supported by the *propositional representations*: if 2.1÷?=8.4 and 2.1×4=8.4 so 2.1×4=2.1÷1/4=8.4. A sequence of true statements leads to the correct answer.

The word problem presented in Table 2 (task 6) can be solved by using the expression 22.5÷1/2. Eva’s strategy was: “It is 45 baskets. I only multiply 22.5 by 2”. Questioned why she multiplied by 2, she answered: “Because of ½. To get the unit we must add five tenths twice, so we multiply by 2”. This was used to discuss division sense when using a divisor less than 1, since the quotient gets bigger then the dividend, and not smaller as it happens with whole numbers. Eva used a *numerical relationship* strategy (relations between operations and part-hole relationship). She understood that two halves make a unit and explained it using two equivalent representations (1/2 and 0.5). This gives meaning to the multiplication by 2 instead of the division by 1/2. Understanding this, she supported her strategy in a *propositional representation* (if 1/2=0.5 and 0.5×2=1, so, 22.5÷1/2=22.5×2=45). The meaning of the “invert and multiply” rule was discussed using Eva´s explanation.

In task 10, the last task of the teaching experiment, Rui computed “$0.75 \div ? = 3$” explaining: “I changed 75 hundredths to 3/4 because it’s equivalent. So, 3/4 divided by a number that I don’t know to get 3… 3/4 dividing by 1/4 is equivalent to 3.3 divided by 1 is equivalent to 3 and 4 divided by 4 is equivalent to 1”. He changed representation from decimals to fractions (*numerical relationships*) and this was essential to see that there is a multiplicative relationship between dividend and quotient (3/4÷?=3). He was not explicit about this relation, but when he stated “3/4 dividing by 1/4” he assumed that the result is 1/4, and this could come from 3×1/4=3/4. To validate his strategy, Rui solved the operation (3/4÷1/4) after knowing the result 3. Since the denominators are equal he divided numerators and denominator to get the answer 1/4. His strategy could be supported by a *propositional representation* that gives meaning to his way of thinking (if 0.75=3/4 so 0.75÷?=3 is equivalent to 3/4÷?=3. If 3×1/4=3/4, then, 3/4÷1/4=3 because 3÷1/4÷4=3/1). Interestingly, Rui always used the word “equivalent” to talk about the equal sing. This shows that he understood it as a sign of equivalence and not as a sign that requires a solution.

Discussion and conclusion

In this paper we share and analyze the most representative mental computation strategies in division of rational numbers, used by students in CI and CII of our study. To develop students mental computation strategies, we designed tasks that include benchmarks, different rational number representations (Kilpatrick et al., 2001), different levels of cognitive demand and contexts to support students to connect number sentences with world problems and vice-versa (e.g., Sinicrope et al., 2002). This allows students to give meaning to number representations and extend previous knowledge about whole numbers.

In the beginning of the study (task 2) most students use memorized rules such as "invert and multiply" two divide two fractions. This is a procedure that most students use, certainly without meaning, because it does not seem to be associated with division (Siebert, 2002). To divide fractions, students multiply most of the time without relating these two operations. When challenged to compute mentally in open number sentences they use numerical relationships strategies, with an emphasis on the change of representation. Decimal division was more difficult for students than fraction division, probably because they learned a rule to divide fractions and not to divide decimals. Therefore, students prefer to change decimals to fractions to use the rule or to relate division and multiplication (as did Maria and Rui). In open number sentences it is not possible to use directly the "invert and multiply" rule. The use of this kind of task in the teaching experiment was an important step in the development of students' mental computation strategies in division, since they need to relate numbers and operations instead of applying rules, where mental representations play an important role. Mental representations as propositional representations support students' reasoning while several relationships are made, and mental models can provide real-life contexts with meaning for students, so they can solve mathematical expressions (as did António). During the teaching experiment, several word problems were used and related with number sentences to help students to create mental representations and make connections between real-life contexts and mathematical expressions (Siebert, 2002). For example, in task 5 we used the expression $12.2 \div 0.5$ to relate later with the problem shown in task 6 (see Table 2). The use of word problems was another important step in the development of students' strategies as it gives a real-life context where students need to search for an operation to solve it. This search allows students to find mathematical expressions where the contexts facilitate an understanding of relationships previously discussed in the classroom. Eva's strategy to solve a word problem in a measurement context represented an opportunity to give meaning to the relationship between dividing by 1/2 and multiplying by 2 as well to the "invert and multiply" rule, used several times by students.

This study shows whenever students apply a memorized rule they mostly apply a procedure without meaning. They explained a set of procedures (as Rita did) where no meaning of division is shared. When students use numerical relationships strategies they use multiplication to solve a division, stressing the relation between these two operations. The inverse relation between these operations emerged in a strong way in students' strategies, especially in open number sentences tasks. The missing value was the divisor that can be calculated dividing dividend and quotient, but students solved it by searching for a known relationship (as Ana did) or the relation between dividend and quotient (as Maria and Rui did). On one side, the use of multiplication can be a sign of students' difficulty in dividing rational numbers, so they search a more familiar operation to solve the problem.

On the other side, the use of multiplication emphasizes multiplicative relationships between numbers and the inverse relation between division and multiplication (e.g., Sinicrope et al., 2002). Partitive division meaning was introduced by António when he used a money context to share equally 3€ by 3 people. He drew his knowledge from division of whole numbers and extended it to rational numbers. He used a known context to model the resolution of a number sentence. António's self-validation of his answer shows a strategy made with understanding. The measurement meaning of division was provided in the basket problem, where Eva's strategy was very useful to give meaning to some relations previously made by students as we already stressed above.

To conclude, this is a singular study in Portugal that provides suggestions to teachers who want to develop students' mental computation strategies in rational numbers division. Sharing and discussing students' strategies with the whole class presents an opportunity for teachers to understand students' reasoning, but also to construct collectively an understanding about the division of rational numbers where several relationships can be explored. Further research focusing mental computation with rational numbers involving different representations is needed, as well as their contributions for the transition between arithmetic and algebra in students learning process.

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